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A Behavioral Approach to Singular Systems

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Abstract. The notion of behaviors introduced by Willems gives a good description of dynamical systems without reference to any particular representation of the system in terms of equations. In this note, we introduce a notion of behaviors that allows us to describe singular systems in a very natural way. The new definition of behaviors given here is closely related to that of a sheaf over the projective line, and we make this connection precise.

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1. Introduction

There has been a great deal of interest in linear systems of the form

$$Gz(k+1) = Fz(k), \qquad w(k) = Hz(k). \tag{1}$$

Here $z \in Z$ is the internal variable, $w \in W$ is the signal variable; F, G are linear maps from Z to the state space X, and H is a linear map from Z to W. This class of systems has been recently studied extensively in the book by Kuijper [5] and in the articles [3, 6, 11, 12]. The following rank conditions are assumed to hold:

$$\operatorname{rk}(sG - F) = \dim X \quad \text{and} \quad \operatorname{rk}\begin{pmatrix} sG - F \\ H \end{pmatrix} = \dim Z.$$

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Following Willems we do not make a distinction between inputs and outputs in the signal space W. Likewise we do not distinguish states and inputs in the space Z.

Notice that a standard singular system

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k) + Du(k), \end{aligned}$$

with $\det(sE - A) \neq 0$ can be written in the form (1) by taking

$$z = \begin{pmatrix} x \\ u \end{pmatrix}, \qquad w = \begin{pmatrix} u \\ y \end{pmatrix}, \qquad G = \begin{pmatrix} E & 0 \end{pmatrix},$$

$$F = \begin{pmatrix} A & B \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}.$$

It should be pointed out that in fact, the class of linear systems of the form (1) is only a bit larger than that of standard linear systems. There is also a strong motivation to study the expanded class of systems, as in (1), because this class gives a smooth compactification of the space of all transfer functions [10].

It is intuitively clear that many properties of the system (1) are determined solely by the first equation in (1). In particular, notions such as controllability of the system and various feedback invariants are determined by the first equation. Thus we believe that, in order to get a better understanding of the original system (1), it is worthwhile studying the first equation independently. So the primary object of our paper are equations of the form

$$Gz(k+1) = Fz(k). \tag{2}$$

We will associate a behavior to such equations. Further, we will show that these new behaviors are in one-to-one correspondence with coherent sheaves on the projective line. In some sense, the main goal of this paper is to make this connection explicit. This connection between sheaves and linear systems has been made several times in the Systems Theory literature, the earliest mention being in the work of Hermann and Martin [8]. This paper arose out of the desire to make this connection precise.

The biggest jump in our paper from the literature in 'Behavioral Systems Theory', is in our expanded definition of 'abstract linear behaviors'. Let us briefly outline the idea of our approach. Consider the following two equations:

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}.$$

If the time axis is $\mathbb{Z}$ it is easily seen that the solution spaces of these equations, that is their behaviors in the sense of Willems [15], consist only of zero trajectory.

On the other hand, over $\mathbb{Z}_+$ the first system does have two linearly independent solutions, namely

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0, 1, 0, \dots \\ 1, 0, 0, \dots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1, 0, 0, \dots \\ 0, 0, 0, \dots \end{pmatrix}.$$

Over $\mathbb{Z}_-$ the second equation also has two linearly independent solutions, namely

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \dots, 0, 1, 0 \\ \dots, 0, 0, 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \dots, 0, 0, 1 \\ \dots, 0, 0, 0 \end{pmatrix}.$$

These two examples suggest that the entire behavior associated to (2) should be the triple (V, V_+, V_-), where V, V_+ and V_- are the solutions over $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{Z}_-$, respectively. We will show in this paper that we can recover our equation from this triple. This means that the triple (V, V_+, V_-) provides a complete description of the given equation.

The behavior triple (V, V_+, V_-) carries a structure which we now want to analyze. First of all the spaces V, V_+ and V_- are closed linear subspaces of sequence spaces equipped with the product topology, and from the purely topological point of view, are what Bourbaki [1] calls topological linear spaces of minimal type. Next, these spaces have canonical operators, namely, the left shift $\sigma\colon V \to V$, the left shift $\sigma_+\colon V_+ \to V_+$ and the right shift $\tau_-\colon V_- \to V_-$. (Notice that the first map σ is invertible and the inverse is the left shift τ.) Further, there are evident restriction maps $r_+\colon V \to V_+$ and $r_-\colon V \to V_-$ which are continuous and are compatible with the left shift and the right shift respectively. Finally notice that, by Willems' completeness property, there are canonical isomorphisms $V \simeq L(V_+, \sigma_+)$ and $V \simeq L(V_-, \tau_-)$, where

$$L(V_+, \sigma_+) = \{(z^{(0)}, z^{(1)}, \dots) \mid z^{(i)} \in V_+ \text{ and } \sigma_+(z^{(i+1)}) = z^{(i)}\}$$

and

$$L(V_-, \tau_-) = \{(z_{(0)}, z_{(1)}, \dots) \mid z_{(i)} \in V_- \text{ and } \tau_-(z_{(i+1)}) = z_{(i)}\}.$$

The above isomorphisms are given respectively by $z \mapsto (z^{(0)}, z^{(1)}, \dots)$, where $z^{(k)} = z|_{[-k,+\infty)} = (z_{-k}, z_{-k+1}, \dots)$ and $z \mapsto (z_{(0)}, z_{(1)}, \dots)$, where $z_{(k)} = z|_{(-\infty,k]} = (\dots, z_{k-1}, z_k)$.

Abstracting the above structure, we obtain what we call an abstract linear behavior.

Throughout the paper we work with a fixed ground field $\mathbb{F}$ which may be the field of real numbers or the field of complex numbers, with the usual topology, or an arbitrary discrete field, in particular a finite field.

2. Spaces of Minimal Type and Shift Spaces

We start with the definition of the 'spaces of minimal type':

DEFINITION 2.1. A topological vector space X over $\mathbb{F}$ is called a *space of minimal type* if it is isomorphic to $\mathbb{F}^I$ with the product topology, for some index set I.

In the case, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, this concept is the same as that defined in Bourbaki [1, Ch. II, §6, Exc. 13]. In the cases of discrete fields, this is what Lefschetz calls a linearly compact space [4, Ch. II, §10, Sec. 9].

We will recall the main properties of such spaces that we will use. Firstly, there is a duality between vector spaces and topological vector spaces of minimal type, given by associating to a space of minimal type V the linear space $V' = \mathrm{Hom}_{\mathbb{F}}^{\mathrm{cont}}(V, \mathbb{F})$ of all continuous linear functionals on V. The inverse of this association is obtained by associating to a given vector space E its dual $E^* = \mathrm{Hom}_{\mathbb{F}}(E, \mathbb{F})$, equipped with the pointwise topology. Secondly, it can be shown, that a continuous linear map $f\colon V \to W$, between two spaces of minimal type is a closed map. Finally, a closed subspace of a space of minimal type is also of minimal type. For a proof of these properties, see the references cited above.

DEFINITION 2.2. A *shift space* S is a pair (V, μ) where V is a space of minimal type and μ is a continuous linear mapping from V into itself.

The easiest examples of shift spaces are $(\mathbb{F}^q)^{\mathbb{Z}}$, $(\mathbb{F}^q)^{\mathbb{Z}_+}$ and $(\mathbb{F}^q)^{\mathbb{Z}_-}$, with the obvious shift operators. More generally, any closed, shift invariant subspace of such spaces are also shift spaces.

To any shift space S one can associate, in a natural way, a module over the ring $\mathbb{F}[s]$ of polynomials with coefficients in $\mathbb{F}$, consisting of the vector space V, together with an action of polynomials defined by by $r(s)\colon v \mapsto r(\mu)v$. We shall denote this associated module also by S. The dual V' can be given the structure of an $\mathbb{F}[s]$-module as well, by defining

$$(r(s)(v'))(v) := v'(r(\mu)(v)), \qquad \text{for all } v \in V.$$

The vector space V' with this module structure will be denoted by S'. The module S' may be 'nicer' than S itself, as shown in the following example.

EXAMPLE 2.3. Let S be $(\mathbb{F}^q)^{\mathbb{Z}_+}$ together with the left shift map. Then S as a $\mathbb{F}[s]$-module is isomorphic to $\mathbb{F}[[s^{-1}]]^q$ which is not a finitely generated $\mathbb{F}[s]$-module. However, S' as a module over $\mathbb{F}[s]$ is isomorphic to the free module $\mathbb{F}^q[s]$. This relationship was also pointed out specifically by Willems [15, second proof of Theorem 5].

One can define morphisms between shift spaces in an obvious manner, namely given two shift spaces, $S_1 = (V_1, \mu_1)$ and $S_2 = (V_2, \mu_2)$, a morphism f from S_1 to

S_2 is a continuous linear map from V_1 to V_2 such that $f\mu_1 = \mu_2 f$. It is easy to see that the map f induces an $\mathbb{F}[s]$-module homomorphism f': $S_2' \to S_1'$.

DEFINITION 2.4. A shift space S is said to be *of finite type* if there exists an injective morphism from S to the space $((\mathbb{F}^q)^{\mathbb{Z}_+}, \sigma_+)$, or equivalently, if the dual module S' is finitely generated.

Among the examples of shift spaces given earlier, $(\mathbb{F}^q)^{\mathbb{Z}}$ is not of finite type, whereas, $(\mathbb{F}^q)^{\mathbb{Z}_+}$ and $(\mathbb{F}^q)^{\mathbb{Z}_-}$ are of finite type.

Note that Fliess [2, p. 228] actually defines a 'linear system' as a finitely generated $\mathbb{F}[s]$-module. Since $\mathbb{F}[s]$ is a principal ideal domain, every finitely generated module M over $\mathbb{F}[s]$ can be represented through a short exact sequence $0 \to \mathbb{F}^p[s] \to \mathbb{F}^q[s] \to M \to 0$; in other words, M can be written as the quotient of the free module $\mathbb{F}^q[s]$ by the module generated by some polynomial matrix $R(s)$ of size $p \times q$. If $M = S'$ is obtained from a shift space S as above, the polynomial matrix $R(s)$ can be viewed as the AR representation of S.

Given a module M over $\mathbb{F}[s]$, it can be regarded as a pair consisting of the underlying vector space and a linear map, corresponding to multiplication by s.. Thus, one can define its dual M^* which is a shift space. Moreover, given an $\mathbb{F}[s]$-homomorphism f: $M_1 \to M_2$ between two modules, one gets a morphism of shift spaces f^*: $M_2^* \to M_1^*$.

DEFINITION 2.5. A shift space $S = (V, \mu)$ is called a *Laurent shift space* if the mapping μ is bijective.

Since a continuous map of minimal spaces is closed, it follows that if $S = (V, \mu)$ is a Laurent shift space, then so is $S^{\mathrm{o}} := (V, \mu^{-1})$ ('o' for 'opposite'). Note that a Laurent shift space can be considered as a module over $\mathbb{F}[s, s^{-1}]$.

An example of a Laurent shift space is $(\mathbb{F}^q)^{\mathbb{Z}}$ with either of the shift operators. Any closed, shift invariant subspace of it is also a Laurent shift space.

We think of shift spaces, as behaviors over a half line, either $\mathbb{Z}_+$ or $\mathbb{Z}_-$. On the other hand, Laurent shift spaces correspond to behaviors over $\mathbb{Z}$. It will be important to associate a behavior over the whole line, to any behavior over a half line. Thus, in our language, we need a procedure that will associate a Laurent shift space to a given shift space. We achieve this by what we call *Laurentization*.

To every shift space $S = (V, \mu)$ we associate a Laurent shift space $L(S) = (\tilde{V}, \tilde{\mu})$ in the following way. The space $\tilde{V}$ consists of all sequences $(v_0, v_1, v_2, \ldots)$, $v_i \in V$ such that $\mu v_{i+1} = v_i$ for all $i \geqslant 0$. The mapping $\tilde{\mu}$ is defined by

$$\tilde{\mu}\colon (v_0, v_1, v_2, \ldots) \mapsto (\mu v_0, \mu v_1, \mu v_2, \ldots) = (\mu v_0, v_0, v_1, \ldots).$$

It is easily verified that $L(S)$ is indeed a Laurent shift space. This will be called the *Laurentization* of S.

EXAMPLE 2.6. Let V be a closed linear shift-invariant subspace of $(\mathbb{F}^q)^{\mathbb{Z}_+}$ and let μ denote the left shift. Consider an element $(v_0, v_1, \ldots)$ of $\tilde{V}$. The module element

$v_0 \in V$ is itself a sequence of elements of vectors; write $v_0 = (w_0, w_1, \ldots)$. Because $\sigma v_1 = v_0$, we have $v_1 = (w_{-1}, w_0, w_1, \ldots)$ for some $w_{-1} \in \mathbb{F}^q$, which moreover must be such that $v_1 \in V$. Continuing like this, we find that there must be elements $w_{-1}, w_{-2}, \ldots$ such that, for all $k \geqslant 0$,

$$v_k = (w_{-k}, w_{-k+1}, \ldots, w_{-1}, w_0, w_1, \ldots) \in V.$$

So $\tilde{V}$ may be identified with the space of all vector sequences $\tilde{v}$ in $(\mathbb{F}^q)^{\mathbb{Z}}$ such that all the right halves of $\tilde{v}$ belong to V.

Remark 2.7. One can show that $L(S) \simeq \mathrm{Hom}_{\mathbb{F}[s]}(\mathbb{F}[s, s^{-1}], S)$, where the latter is equipped with the pointwise topology.

The following lemma will be important in Section 4.

LEMMA 2.8.

(i) *If S is a shift space, then there is a canonical isomorphism*

$$(L(S))' \simeq S' \otimes \mathbb{F}[s, s^{-1}].$$

(ii) *If M is a $\mathbb{F}[s]$-module, then there is a canonical isomorphism*

$$L(M^*) \simeq (M \otimes \mathbb{F}[s, s^{-1}])^*.$$

Proof. If $S = (V, \mu)$ is a shift space, then by definition $L(S)$ is the inverse limit of the sequence

$$V \overset{\mu}{\longleftarrow} V \overset{\mu}{\longleftarrow} V \cdots.$$

If M is a $\mathbb{F}[s]$-module then $M \otimes \mathbb{F}[s, s^{-1}]$ can be viewed as the direct limit of the sequence

$$M \overset{\times s}{\longrightarrow} M \overset{\times s}{\longrightarrow} M \cdots,$$

where $\times s$ denotes the multiplication by s. Applying the functor $'$ to the first sequence will result in the first isomorphism. Applying the functor * to the second sequence will result in the second isomorphism. □

3. Abstract Linear Behaviors

As illustrated by the examples in the introduction, the modeling of linear dynamic phenomena sometimes calls for more modeling power than can be delivered by the shift spaces discussed above. In this section we shall introduce abstract behaviors which tie together 'past' trajectories, 'future' trajectories, and trajectories 'for all time'.

DEFINITION 3.1. An *abstract (linear) behavior* is a five-tuple

$$\mathcal{B} = (S_-, S_+, S, r_-, r_+)$$

satisfying the following axioms:

(i) $S_- = (V_-, \tau_-)$, the space of 'past' trajectories, and $S_+ = (V_+, \sigma_+)$, the space of 'future' trajectories, are shift spaces of finite type,
(ii) $S = (V, \sigma)$, the space of trajectories 'for all time', is a Laurent shift space,
(iii) $r_-\colon (V, \sigma^{-1}) \to S_-$ and $r_+\colon S \to S_+$ are morphisms,
(iv) the canonical extensions of r_- and r_+ to maps $\tilde{r}_-\colon (V, \sigma^{-1}) \to L(S_-)$ and $\tilde{r}_+\colon S \to L(S_+)$ are isomorphisms.

We think of elements of V_-,V_+ and V as trajectories over $\mathbb{Z}_-$, $\mathbb{Z}_+$ and $\mathbb{Z}$, respectively. The maps r_- and r_+ are regarded as the maps corresponding to restricting trajectories on $\mathbb{Z}$ to half-lines. The last condition in our definition should be interpreted as saying that the space of trajectories obtained by extending the trajectories defined on the half lines to all of $\mathbb{Z}$, is the same as the space of trajectories given on $\mathbb{Z}$.

Remark 3.2. We can think of V_+ as the space of trajectories, not only on $\mathbb{Z}_+$, but also on the intervals $[t, \infty)$ for any t, since the behavior is time-invariant. We can then think of $r_+\sigma^t$ as restricting a trajectory on $\mathbb{Z}$ to $[t, \infty)$.

EXAMPLE 3.3. The easiest example of a linear behavior is the following. Let σ and σ_+ denote the left shifts on $\mathbb{F}^{\mathbb{Z}}$ and $\mathbb{F}^{\mathbb{Z}_+}$ respectively, and let τ_- denote the right shift on $\mathbb{F}^{\mathbb{Z}_-}$. Let $S = (\mathbb{F}^{\mathbb{Z}}, \sigma)$, $S_+ = (\mathbb{F}^{\mathbb{Z}_+}, \sigma_+)$, $S_- = (\mathbb{F}^{\mathbb{Z}_-}, \tau_-)$, and let r_- and r_+ be the restriction maps. We shall denote this behavior by $\mathcal{C}$. The behavior that is obtained by replacing $\mathbb{F}$ in this example by a finite-dimensional vector space X over $\mathbb{F}$ will be denoted by $\mathcal{C} \otimes X$ (it can indeed be obtained as the suggested tensor product).

The following two examples show that our behaviors include Willems' behaviors on $\mathbb{Z}$ and $\mathbb{Z}_+$.

EXAMPLE 3.4. Let V be a closed linear shift-invariant subspace of $(\mathbb{F}^q)^{\mathbb{Z}}$; so V is a complete linear time-invariant behavior on $\mathbb{Z}$ in the terminology of Willems. To such a behavior one can associate an abstract behavior in the following way. Let V_+ consist of all right-infinite sequences $(w_0, w_1, \ldots)$ that are right halves of elements of V. Let V_- consist of all left-infinite sequences $(\ldots, w_{-1}, w_0)$ that are left halves of elements of $\mathcal{B}$. The maps σ and σ_+ are the right shifts and τ_- is the left shift. Finally, let $r_+\colon (V, \sigma) \to (V_+, \sigma_+)$ and $r_-\colon (V, \sigma^{-1}) \to (V_-, \tau_-)$ be the obvious restriction mappings. Using the completeness property of V, one can show that the five-tuple defined in this way is indeed an abstract behavior.

EXAMPLE 3.5. Now let V_+ be a closed linear shift-invariant subspace of $(\mathbb{F}^q)^{\mathbb{Z}}_+$. To associate an abstract behavior to this, one can proceed as follows. Define V to be the Laurentization of V_+, as in Example 2.6. Now, define the rest of the data for the behavior as in the previous example, with V being the behavior on $\mathbb{Z}$.

DEFINITION 3.6. A *morphism* from an abstract behavior $\mathcal{B} = (S_-, S_+, S, r_-, r_+)$ to an abstract behavior $\bar{\mathcal{B}} = (\bar{S}_-, \bar{S}_+, \bar{S}, \bar{r}_-, \bar{r}_+)$ is a pair of mappings (f_-, f_+), with $f_-: S_- \to \bar{S}_-$, and $f_+: S_+ \to \bar{S}_+$, such that they induce the same homomorphism from S to $\bar{S}$.

The following definition will be used later.

DEFINITION 3.7. A morphism $\phi = (f_-, f_+)$ between two behaviors $\mathcal{B}$ and $\bar{\mathcal{B}}$ is said to be *almost injective*, if the kernels of the maps f_- and f_+ are finite-dimensional vector spaces.

Willems' definitions of controllability and autonomicity can be adapted to our notion of behaviors very easily. Following Willems' we take controllability to mean that any two trajectories w_- and w_+ can be joined with an unspecified piece of length n between them. Thus the notion of controllability can be formalized as follows:

DEFINITION 3.8. Let $\mathcal{B} = (S_-, S_+, S, r_-, r_+)$ be an abstract behavior. Then $\mathcal{B}$ is said to be controllable, if given any $w_- \in S_-$ and $w_+ \in S_+$, there exists a $w \in S$ such that $r_-(w) = w_-$ and $r_+(\sigma^n(w)) = w_+$ for some n.

On the other hand, in an autonomous behavior every trajectory defined on $\mathbb{Z}$ is uniquely determined by its past. Thus in our context, we have the following:

DEFINITION 3.9. Let $\mathcal{B} = (S_-, S_+, S, r_-, r_+)$ be an abstract behavior. Then $\mathcal{B}$ is said to be autonomous, if the map r_- is injective.

Finally, we introduce the important notion of *real* behaviors. Having in mind Remark 3.2 the following definition says that a behavior is *real*, if a trajectory on $\mathbb{Z}$ is determined uniquely by its restrictions on $(-\infty, 0]$ and $[1, \infty)$.

DEFINITION 3.10. An abstract behavior $\mathcal{B}$ is said to be *real* if there are no nonzero elements of V that satisfy both $r_- v = 0$ and $r_+ \sigma v = 0$.

The above definition should clarify the fact that our definition of abstract behaviors is much wider than the standard notion of behaviors. But this expanded notion of behaviors is needed in order to have a one-to-one correspondence between sheaves and behaviors demonstrated in the next section. Of course, the notion of real behavior is in itself, wider than the standard notion of behaviors.

4. Relation to Coherent Sheaves

The link between behaviors and sheaves on $\mathbb{P}^1$ was alluded to by Willems in [15], p. 574. While all the behaviors considered in *op. cit.* can be seen to be coherent sheaves on $\mathbb{P}^1$, the converse is not true. With our definition of abstract linear behaviors, we will see that there is a one-to-one correspondence between abstract behaviors and coherent sheaves on $\mathbb{P}^1$.

The projective line $\mathbb{P}^1$ over $\mathbb{F} = \mathbb{C}$ is most familiar as the Riemann sphere obtained by adjoining the point at infinity to the complex plane. For a general field $\mathbb{F}$, $\mathbb{P}^1$ can be thought of as the space of all lines in $\mathbb{F}^2$. As an algebraic manifold, $\mathbb{P}^1$ can be covered by two open sets, both isomorphic to $\mathbb{F}$, but glued together by identifying the point $s \in \mathbb{F}$ in one copy of $\mathbb{F}$ with the point $t = s^{-1}$ in the other copy of $\mathbb{F}$.

In general, a sheaf of modules over an algebraic variety can be described as a collection of modules each associated with an open subset of the variety, tied together by certain restriction mappings and compatibility rules. When the algebraic variety considered is the projective line, one can make do with just using the two basic open subsets and their intersection. Formally then, a *coherent sheaf* over $\mathbb{P}^1$ consists of a quintuple (M_-, M_+, M, i_-, i_+), where M_- is a module over $\mathbb{F}[t]$, M_+ is a module over $\mathbb{F}[s]$ and M is a module over the ring $\mathbb{F}[s, t]$, where s and t satisfy the relation $st = 1$. Further, i_-: $M_- \to M$ and i_+: $M_+ \to M$ are homomorphisms, such that the induced maps from $M_- \otimes_{\mathbb{F}[t]} \mathbb{F}[s, t]$ and $M_+ \otimes_{\mathbb{F}[s]} \mathbb{F}[s, t]$ to M are isomorphisms. There is an obvious notion of morphisms between sheaves.

The simplest example of a sheaf on $\mathbb{P}^1$, is the quintuple $(\mathbb{F}[t], \mathbb{F}[s], \mathbb{F}[s, t], i_-, i_+)$, where i_- and i_+ are the inclusion maps. This sheaf is called the structure sheaf of $\mathbb{P}^1$ and is denoted by $\mathcal{O}_{\mathbb{P}^1}$.

Given a sheaf $\mathcal{G} = (M_-, M_+, M, i_-, i_+)$ and an integer n one defines the twist of $\mathcal{G}$ by n as the sheaf $\mathcal{G}(n) = (M_-, M_+, M, i_-, s^n i_+)$. Given a sheaf $\mathcal{G}$ and a finite-dimensional vector space X, then tensoring all of the components of $\mathcal{G}$ by X, one gets the sheaf $\mathcal{O}_{\mathbb{P}^1} \otimes X$.

In algebraic geometry, one says that a sheaf is *locally free* if its modules are free and a sheaf is *finite* if the modules are of finite length. A sheaf $\mathcal{G}$ is said to be *generated by global sections*, if there exists a surjective morphism from q copies of $\mathcal{O}_{\mathbb{P}^1}$ onto $\mathcal{G}$, for some integer q.

Given an abstract behavior $\mathcal{B} = (S_-, S_+, S, r_-, r_+)$, we let $M_+ = S'_+$, $M_- = S'_-$ and $M = S'$. The maps i_- and i_+ are the dual maps r'_- and r'_+. By Lemma 2.8, the maps i_- and i_+ satisfy the compatibility conditions to give us a sheaf. Conversely, given a coherent sheaf $\mathcal{G} = (M_-, M_+, M, i_-, i_+)$, we let $S_- = M^*_-$, $S_+ = M^*_+$ and $S = M^*$. The maps r_- and r_+ are the dual maps i^*_- and i^*_+. Again by Lemma 2.8, the maps r_- and r_+ satisfy the compatibility condition for behaviors.

THEOREM 4.1. *The correspondence between abstract behaviors and coherent sheaves on $\mathbb{P}^1$ described above is one-to-one.*

For example, the behavior $\mathcal{C}$ and the sheaf $\mathcal{O}_{\mathbb{P}^1}$ correspond to each other.

Remark 4.2. It is interesting to note that under this correspondence, controllable behaviors correspond to locally free sheaves, autonomous behaviors to finite sheaves, and real behaviors to sheaves generated by global sections.

If $\mathcal{G} = (M_-, M_+, M, i_-, i_+)$ is a sheaf, then one defines its 0-*dimensional cohomology space*, $H^0(\mathcal{G}, \mathbb{P}^1)$ as the kernel of the following canonical $\mathbb{F}$-linear map

$$M_- \oplus M_+ \to M, \qquad (m_-, m_+) \mapsto i_+(m_+) - i_-(m_-).$$

The space $H^1(\mathcal{G}, \mathbb{P}^1)$ is defined to be the cokernel of this map. The following is a standard fact in algebraic geometry, which we will use in the next section: *The cohomologies of any coherent sheaf are finite-dimensional vector spaces.*

5. Realization Theory

Let us call a minimal pencil any quadruple (X, Z, F, G) which satisfies the condition $\mathrm{rk}(sG - F) = \dim X$.

In the introduction, we have associated a behavior to a minimal pencil (X, Z, F, G) by taking trajectories of the equation $Gz(k+1) = Fz(k)$. This behavior will be called the behavior of the pencil (X, Z, F, G). Clearly, the behavior of a minimal pencil is real.

In this section, we will show that given a real behavior $\mathcal{B}$, we can canonically construct a minimal pencil, whose behavior is $\mathcal{B}$. We will carry out this construction in two ways: first, in the realm of behaviors alone, and secondly, by working with the sheaf corresponding to a behavior.

DEFINITION 5.1. Let $\mathcal{B}$ be an abstract behavior. We shall say that $v_- \in V_-$ and $v_+ \in V_+$ can be *joined* if there exists $v \in V$ such that $r_- v = v_-$ and $r_+ v = v_+$. We say that $v_- \in V_-$ and $v_+ \in V_+$ can be *concatenated* if there exists $v \in V$ such that $r_- v = v_-$ and $r_+ \sigma v = v_+$.

A word about concatenation. As noted earlier, since the behavior is time invariant, v_+ may be viewed as a trajectory on $[1, \infty)$. With this in mind, to say that $r_+ \sigma v = v_+$, means essentially that $v|_{[1,\infty)} = v_+$.

It is natural to ask, when two trajectories can be joined or concatenated. The answer is given by the vector spaces defined as follows:

DEFINITION 5.2. Let Z be the cokernel of the map

$$\begin{pmatrix} -r_- \\ r_+ \end{pmatrix} : V \to V_- \times V_+$$

and let X be the cokernel of the map

$$\begin{pmatrix} -r_- \\ r_+ \sigma \end{pmatrix} : V \to V_- \times V_+$$

These two vector spaces have finite dimensions. Indeed, let $\mathcal{G} = \mathcal{B}'$. Then, it is easily seen that $H^0\mathcal{G} = Z'$ and $H^0\mathcal{G}(-1) = X'$, and since $H^0\mathcal{G}$ and $H^0\mathcal{G}(-1)$ are finite-dimensional so are Z and X.

We shall denote by 'J_-' the mapping that assigns to an element v_- of V_- the equivalence class of $(v_-, 0)$ in Z; 'J_+' denotes the mapping that assigns to $v_+ \in V_+$ the equivalence class of $(0, v_+)$ in Z. We also introduce mappings 'C_-' from V_- to X and 'C_+' from V_+ to X, which are defined analogously to J_- and J_+ but with Z replaced by X.

Remark 5.3. Two elements $v_- \in V_-$ and $v_+ \in V_+$ can be joined if and only if $J_-v_- = J_+v_+$. Similarly, the two elements v_- and v_+ can be concatenated if $C_-v_- = C_+v_+$. This follows from the definitions.

Intuitively, the vector space Z corresponds to all possible values of the trajectories at a fixed time, say $t = 0$. Given that the behavior is time-invariant, Z can also be identified with possible values at time $t = 1$. Now, the space X can be identified with pairs in $Z \times Z$, for which there does not exist any trajectory in the behavior $\mathcal{B}$, with these values at 0 and 1.

We define maps F and G from Z to X, as follows. The map F assigns the equivalence class of (v_-, v_+) to the equivalence class of (v_-, σ_+v_+), and G assigns it to (τ_-v_-, v_+). So every abstract behavior defines a quadruple (X, Z, F, G). It can be shown that this quadruple satisfies the rank condition. We call this the canonical realization of $\mathcal{B}$.

As an example, it is easy to see that the canonical realization of the behavior $\mathcal{C} \otimes W$ for a finite-dimensional vector space W, is the quadruple $(\{0\}, W, 0, 0)$.

THEOREM 5.4. *The mapping which assigns to a minimal pencil its behavior is one-to-one; its inverse is the mapping that assigns to a real behavior its canonical realization.*

Proof. In view of the one-to-one correspondence between behaviors and coherent sheaves this immediately follows from [6, Theorem 1.1]. □

We would like to illustrate the main point of this paper by giving an alternate proof of Theorem 5.4. As mentioned in the introduction, one of the main points of this paper is to make the connection between abstract behaviors and sheaves on $\mathbb{P}^1$. We wish to illustrate the usefulness of this connection, by giving a sheaf theoretic proof of the above theorem.

Sheaf-Theoretic Proof. First of all, since the behavior $\mathcal{B}$ is assumed to be real, the corresponding sheaf $\mathcal{G}$ is generated by its global sections. Therefore, there is a surjective map

$$H^0(\mathbb{P}^1, \mathcal{G}) \otimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\phi} \mathcal{G} \to 0.$$

The kernel of this map is a locally free sheaf $\mathcal{F}$ and therefore, one has the short exact sequence

$$0 \to \mathcal{F} \to H^0(\mathbb{P}^1, \mathcal{G}) \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{G} \to 0. \qquad (*)$$

The long exact sequence of cohomology groups gives the sequence:

$$0 \to H^0(\mathbb{P}^1, \mathcal{F}) \to H^0(\mathbb{P}^1, \mathcal{G}) \xrightarrow{H^0(\phi)} H^0(\mathbb{P}^1, \mathcal{G}) \to H^1(\mathbb{P}^1, \mathcal{F}) \to 0.$$

Now, the map $H^0(\phi)$ is an isomorphism, therefore, $H^0(\mathbb{P}^1, \mathcal{F}) = H^1(\mathbb{P}^1, \mathcal{F}) = 0$. From here one concludes that $\mathcal{F} \simeq V \otimes \mathbb{P}^1$ for some vector space V. Further, by tensoring the exact sequence $(*)$ by $\mathcal{O}_{\mathbb{P}^1}(-1)$ and taking the long exact sequence of cohomologies as above, one finds that $H^0(\mathbb{P}^1, \mathcal{F}) = V \simeq H^0(\mathbb{P}^1, \mathcal{G}(-1))$. Thus the short exact sequence $(*)$ is of the form:

$$0 \to H^0(\mathbb{P}^1, \mathcal{G}(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\psi} H^0(\mathbb{P}^1, \mathcal{G}) \otimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\phi} \mathcal{G} \to 0.$$

Now, the map ψ can be represented in matrix form, after choosing bases for the two vector spaces $H^0(\mathbb{P}^1, \mathcal{G})$ and $H^0(\mathbb{P}^1, \mathcal{G}(-1))$, by a matrix of the form $sG' - F'$, where F' and G' are scalar matrices. If we now define $X = H^0(\mathbb{P}^1, \mathcal{G}(-1))'$ and $Z = H^0(\mathbb{P}^1, \mathcal{G})'$, then the quadruple (X, Z, F, G) defines the same behavior as $\mathcal{B}$.

Conversely, given a quadruple (X, Z, F, G), the condition $\mathrm{rk}(sG - F) = \dim X$ implies that the map $\psi = sG' - tF'$ from $X' \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ to $Z' \otimes \mathcal{O}_{\mathbb{P}^1}$ is an injective sheaf map. So we can define a sheaf $\mathcal{G}$ to be the quotient sheaf in the short exact sequence:

$$0 \to X' \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\psi} Z' \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{G} \to 0.$$

The behavior associated to the sheaf $\mathcal{G}$ is the same as the behavior associated to the quadruple (X, Z, F, G). Clearly, the sheaf $\mathcal{G}$ is generated by global sections, therefore, the corresponding behavior $\mathcal{B}$ is real. Thus the two constructions are inverse to each other and provide a one-to-one correspondence between real behaviors and quadruples as claimed in the theorem.

6. Linear Systems

Let W be a finite-dimensional linear space (a signal space).

The starting point of the paper was the linear system (1), that consisted of the quintuple (X, Z, F, G, H), with the conditions

$$\mathrm{rk}(sG - F) = \dim X \quad \text{and} \quad \mathrm{rk}\begin{pmatrix} sG - F \\ H \end{pmatrix} = \dim Z.$$

We shall call such a linear system a state space linear system. As noted in the previous section, the minimal pencil (X, Z, F, G) gives rise to a behavior $\mathcal{B}$. The map H determines a morphism from $\mathcal{B}$ to the behavior $\mathcal{C} \otimes W$. The second rank condition implies that this map is almost injective.

DEFINITION 6.1. A *behavioral linear system* (with signal space W) is a pair $(\mathcal{B}, \theta)$, where $\mathcal{B}$ is a behavior and θ is an almost injective morphism from $\mathcal{B}$ to the behavior $\mathcal{C} \otimes W$.

In terms of the corresponding sheaves, if $\mathcal{G}$ is the sheaf associated to the behavior $\mathcal{B}$, then the linear system $(\mathcal{B}, \theta)$ corresponds to giving the sheaf $\mathcal{G}$ along with a map θ^* from $W' \otimes \mathcal{O}_{\mathbb{P}^1}$ to $\mathcal{G}$. The fact that θ is almost injective, implies that the sheaf map θ^* is *generically surjective*.

We have already associated a behavioral linear system to a state space linear system. Call it the behavioral representation.

Suppose we are given a behavioral linear system $(\mathcal{B}, \theta)$. The corresponding map θ^* of sheaves is almost surjective, and this implies that the sheaf $\mathcal{G}$ associated to the behavior $\mathcal{B}$ is generated by global sections. Hence, the behavior $\mathcal{B}$ is real. Therefore, by the realization theorem of the previous section, we can associate to it a minimal pencil (X, Z, F, G). As seen above one has the map θ^*: $W' \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{G}$. This map gives the map $H' = H^0(\theta^*)$: $W' \to H^0(\mathbb{P}^1, \mathcal{G}) = Z'$. The condition that θ is almost injective, implies that the quintuple (X, Z, F, G, H) is a state space linear system. We call it the canonical realization of the behavioral linear system.

THEOREM 6.2. *The mapping which assigns to a state space linear system its behavioral representation is one-to-one. Its inverse is the mapping that assigns to a behavioral linear system its canonical realization.*

Proof. Follows from Theorem 5.4. □

Concluding, we define controllability and observability as follows.

DEFINITION 6.3. A linear system $(\mathcal{B}, \theta)$ is said to be controllable if $\mathcal{B}$ is controllable, and observable if θ is injective.

A homogeneous behavior, as defined in [12] corresponds to an observable linear system, according to the above definitions.

7. Conclusions

In this paper we introduced a notion of linear behaviors that is defined independently of an embedding into some space of sequences. We showed that there is a one to one correspondence between coherent sheaves over the projective line and the class of abstract behaviors as introduced in this paper. We also identified a subclass of abstract behaviors for which a realization in a first-order form can be written down.

The fact that every linear time invariant system can be viewed as a coherent sheaf over the projective line was first observed by Martin and Hermann [8] but in general not every coherent sheaf defines a linear time invariant system in the traditional sense. The connection between coherent sheaves and linear behaviors has first been worked out in more detail by the first author in [6, 7]. The correspondence with coherent sheaves extends a well known duality between 'concrete AR-systems' on one side and a set of quotient modules on the other side. This duality has also been worked out in other directions, for instance in the work of

Fliess (as summarized for instance in [2]), and in a study of 2D systems by Rocha and Willems [13]. A study of great generality was undertaken by Oberst [9], but an integration of all aspects has not yet been accomplished. It appears that linear system theory finds itself at the crossroads of a number of key mathematical ideas, with connections that become particularly clear within the behavioral context.

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